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Directed Polymers in a Random Environment: Some Results on Fluctuations

M. S. T. Piza[1]

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We consider a polymer model on $\mathbb{Z}_+^d$ where to each edge e is associated a random variable $v(e)$. A polymer configuration is represented by a directed path r and has a weight $\exp[-\beta \sum_{e \in r} v(e)]$, with $\beta = 1/T$ the inverse temperature. We extend some rigorous results that have been obtained for the ground state of this model to finite temperatures. In particular we obtain some upper and lower bounds on sample-to-sample free energy fluctuations, and also rigorous scaling inequalities between the exponents describing free energy fluctuations and transversal displacements of polymer configurations.

KEY WORDS: Directed polymers; random environment; scaling relations; free energy; variance.

1. INTRODUCTION

Directed polymers in a random environment have been a subject of great interest in recent years, representing one of the simplest models where one finds a low temperature phase in which the quenched disorder has a non-perturbative effect on determining the behavior of the system.[1-7] Despite the progress made in the heuristic understanding of such models, much less has been done on the rigorous side. In this work we extend some rigorous results that have been obtained for the ground state of one class of directed polymer models[8-11] to nonzero temperature.

The directed polymer model we are going to consider is defined as follows. Initially, to each edge $e = (\vec{u}, \vec{v})$, between nearest neighbor sites $\vec{u}$ and $\vec{v}$ of $\mathbb{Z}_+^d$, we attach a non-negative random variable $v(e)$ (the "potential" associated with the edge e). We consider the simplest situation where

[1] Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB, United Kingdom.

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these random variables are independent and identically distributed (i.i.d.) with common distribution function $G(x)$ and define for a path r of length n (a sequence of nearest neighbor sites $\vec{v}_0, \vec{v}_1,..., \vec{v}_n$ and edges $e_i = (\vec{v}_{i-1}, \vec{v}_i)$), the energy of r as

$$E(r) = \sum_{i=1}^{n} v(e_i) \tag{1.1}$$

We now restrict ourselves to *directed paths*, that is, we are allowed to travel only along paths whose coordinates never decrease and for a given directed path r from the origin $\vec{0}$ to a point $\vec{x}$ in $\mathbb{Z}^d_+$, assign a probability

$$\mu^{\vec{x}}_{\beta}(r; \omega) = \frac{\exp[-\beta E(r; \omega)]}{Z^{\vec{x}}_{\beta}(\omega)} \tag{1.2}$$

Here ω denotes a given realization of the random potentials (so $\mu^{\vec{x}}_{\beta}(r)$ is a measure-valued random variable on the underlying probability space where the potentials are defined) and the partition function

$$Z^{\vec{x}}_{\beta}(\omega) = \sum_{r: \vec{0} \to \vec{x}} \exp[-\beta E(r; \omega)] \tag{1.3}$$

is the normalization factor. With the above definitions we have a model for a (directed) polymer in a random environment determined by a given realization of the potentials and (1.2) determines the probability of a configuration of the polymer having energy $E(r)$ at temperature $T = 1/\beta$. The free energy of the system is given as usual by

$$F^{\vec{x}}_{\beta}(\omega) = -\frac{1}{\beta} \log Z^{\vec{x}}_{\beta}(\omega) \tag{1.4}$$

The ground state of the model is obtained by taking the limit of zero temperature ($\beta \to \infty$). In this limit we have a measure supported on the set of paths with lowest energy with the ground state energy given by

$$E^{\vec{x}}_{gs}(\omega) = \inf\{E(r): r \text{ a directed path from } \vec{0} \text{ to } \vec{x}\} \tag{1.5}$$

The optimization problem posed by (1.5) is a version of a model also known as first-passage percolation model in the probability literature (see ref. 12 for a review). Since here we are considering directed paths, we obtain what is usually called a directed or oriented first-passage percolation model. One may also consider the situation where the polymer is only pinned at the origin but with its length (i.e., the number of edges) fixed. In this case we denote the corresponding measure on the space of directed paths of length n by

$$\mu_\beta^{f,n}(r;\omega) \equiv \frac{\exp[-\beta E(r;\omega)]}{Z_\beta^{f,n}(\omega)} \tag{1.6}$$

with free energy

$$F_\beta^{f,n}(\omega) = -\frac{1}{\beta} \log Z_\beta^{f,n}(\omega) \tag{1.7}$$

In general we will express thermal averages (i.e., averages with respect to the (random) Gibbs measures μ_β) by $\langle \cdot \rangle$ and averages with respect to the random environment (realizations of the potential) by $\mathbb{E}(\cdot)$. From now on we will assume, unless otherwise stated, that the $v(e)$'s have a finite second moment ($\mathbb{E}(v^2(e)) < \infty$).

Two quantities of interest in this model (which, as will be discussed shortly, turn out to be related to each other) are the large n behavior of the fluctuations of the free energy and the behavior of typical polymer configurations (w.r.t. the measures (1.2) and (1.6)). In particular, it is expected that, for large n,

$$\text{var}(F_\beta^{f,n}) \equiv \mathbb{E}[(F_\beta^{f,n})^2] - \mathbb{E}^2(F_\beta^{f,n}) \sim n^{2\chi} \tag{1.8}$$

with χ depending on β and the dimension d.

The conjectured picture is as follows (see ref. 13 for more details and additional references). For $d > 3$, and under some conditions on G (for example, G continuous and with an exponential tail should be more than enough), the system is expected to undergo a transition from a high temperature (low disorder) regime where the fluctuations of the free energy are of order 1 (so that $\chi = 0$), to a low temperature (high disorder) regime, where $\chi > 0$, with the fluctuations in this regime being governed by the zero temperature exponent. For $d \leqslant 3$ (and again, under some conditions on the distribution G) it is conjectured that the fluctuations are determined by the zero temperature exponent for all $T < \infty$, hence the behavior of the system is dominated by the disorder at all temperatures.

A quantity of related interest is the behavior of typical paths with respect to the measure (1.6). In the infinite temperature/zero disorder limit we have $\mu_\beta^{f,n} \to \mu_0^{f,n}$ weakly, with probability one. Here $\mu_0^{f,n}(\cdot)$ is a measure which assigns equal probability to all directed paths of length n. The distribution of polymer configurations is independent of the random environment and we have trivially a diffusive behavior that is, typical paths of length l will have deviations of order $l^{1/2}$ about the diagonal line (i.e., the line through the origin in the direction of the vector $\vec{e}_{diag} = (1, 1,..., 1)$). Alternatively, denoting by $\vec{r}_k$ the position of the path after k steps, $\vec{r}_k - (k/d)\,\vec{e}_{diag}$ can be seen as a random walk on the $d-1$ dimensional hyperplane through the origin, and orthogonal to $\vec{e}_{diag}$. The absence of large fluctuations, for $d \geqslant 4$, makes possible the use of perturbative methods to show that the diffusive behavior of typical paths (or equivalently, of the $d-1$ dimensional random walk) persists for sufficiently high temperature or low disorder. Indeed it has been proven, for a slightly different model than the one studied here,[14,15] that for high enough temperatures (low enough disorder) typical paths have the same behavior as in the infinite T/no disorder situation. In fact one can show, by the same method used by Bolthausen in ref. 15 (see also refs. 20, 21, 22), that for $d \geqslant 4$, there exists a constant $\rho = \rho(d) < 1$ such that if $\mathbb{E}[\exp - 2\beta v(e)]/\mathbb{E}^2[\exp - \beta v(e)] < 1/\rho$ then

$$\lim_{n \to \infty} \frac{1}{n^{2\zeta}} \left[\mu_\beta^{f,n} \left(\left| \vec{r}_n - \frac{n}{d} \vec{e}_{diag} \right|^2 \right) \right] = \frac{d-1}{d} \tag{1.9}$$

for almost every realization of the random environment, with the exponent ζ assuming the value 1/2. Also, as in ref. 15, a central limit theorem result can be obtained. Namely, for almost all realizations of the random environment, $(\vec{r}_n - (n/d)\,\vec{e}_{diag})/\sqrt{n}$ converges weakly to a normally distributed random variable with covariance matrix having $(d-1)/d^2$ for its diagonal elements and $-(1/d^2)$ for the off diagonal ones. The exponent ζ introduced above describes the transverse displacement of the polymer about the diagonal. It turns out that χ and ζ are not independent of each other but are believed to obey the scaling relation $\chi = 2\zeta - 1$. This relation has been derived heuristically in several different ways (under, sometimes implicit, assumptions on the distribution of the potential) and for a variety of models for which the asymptotic behavior of the quantities considered here should be the same. Under appropriate assumptions on G. this relation is believed to hold in all dimensions and for all $T < \infty$.[1,4,13] In particular it implies that $\chi > 0$ should be associated with super-diffusive behavior (i.e., $\zeta > 1/2$). It turns out that for $d = 2$ some continuous versions of the polymer model have been shown to be closely related to the Burgers' equation with noise.[3,1,4,5] This connection has been used to obtain an additional (non-rigorous) scaling relation between the two exponents, namely $\zeta = 2\chi$.[1,3,5] Combining these relations one obtains $\chi = 1/3$ and $\zeta = 2/3$ in two dimensions. For $d > 2$ exact values of the exponents are not known but it is believed that χ should decrease with increasing d. On the other hand it seems to be an unsettled issue whether there is an upper critical dimension above which $\chi = 0$ for all temperatures or if $\chi > 0$, therefore $\zeta > 1/2$, in all dimensions, for low enough temperatures (strong enough disorder) (see refs. 1, 4, 28, 29).

In the case where both ends are fixed one should, in principle, allow the above exponents to have a direction dependence, with $\zeta_{\hat{x}}$ describing transverse displacements about the line connecting the origin and the end point, say $n\hat{x}$, and $\chi_{\hat{x}}$ describing the asymptotic behavior of $\text{var}(F_{\beta}^{n\hat{x}})$ as n approaches infinity. It is nevertheless expected that these exponents are independent of the direction $\hat{x}$, (away from some special directions such as the coordinate axes, and under some conditions on the distribution of the random potential; see the discussion after Proposition 3) and that they assume the same values as in the free end case.

In the following section we state our results. Initially we give some bounds on free energy fluctuations. The lower bound shows that in $d = 2$ the free energy fluctuations in fact diverge for all temperatures, while the upper bound holds for all dimensions and can be summarized as $\chi \leqslant 1/2$. Next we state two propositions concerning scaling inequalities between versions of the exponents χ and ζ, which hold at all temperatures. The first one has the form $\chi_{\hat{x}} \geqslant [1 - \zeta_{\hat{x}}(d-1)]/2$, for all $\hat{x}$. The second is a scaling inequality of the form $\zeta_{\hat{x}} \leqslant (1 + \chi')/2$ where χ' is an exponent related to χ. Our last result is an upper bound on the transversal fluctuations of minimizing paths in the free end case, and in the diagonal direction (that is $\hat{x} = \hat{x}_{diag} \equiv \vec{e}_{diag}/\sqrt{d}$) for the fixed end case, under a "curvature assumption." Proofs are presented in Section 3.

2. RESULTS

Our first proposition extends some upper and lower bounds obtained for fluctuations of the ground state energy in the first-passage percolation model[9, 10] to finite temperatures The lower bound shows that under essentially minimal hypotheses on the distribution of the random potential the free energy fluctuations, for $d = 2$, diverge for all temperatures at least logarithmically fast. This of course does not provide any information on the exponent χ. The upper bound shows that for all temperatures and all dimensions one has $\chi \leqslant 1/2$. In what follows $|\vec{x}|$ will denote the Euclidean norm of $\vec{x}$ and $|\vec{x}|_1 = \sum_{i=1}^{d} |x_i|$ the L^1 norm. We will also use the notation $\vec{e}_{diag} \equiv (1, 1,..., 1)$ and $\hat{x}_{diag} \equiv \vec{e}_{diag}/\sqrt{d}$.

Proposition 1. Consider the directed polymer model on $\mathbb{Z}_+^d$ with i.i.d. $v(e)$'s such that $\mathbb{E}(v^2(e)) < \infty$ and $\text{var}(v(e)) > 0$. Then,

(a) For $0<\beta<\infty$ and $d=2$ there exist constants $0<c_1(\beta, \mathrm{G})<\infty$, such that for all $\vec{x} \in \mathbb{Z}_+^2$

$$\operatorname{var}(F_\beta^{\vec{x}}) \geqslant c_1 \log |\vec{x}| \tag{2.1}$$

(b) For $0<\beta<\infty$ there exist constants $0<c_2(\mathrm{G}, d)<\infty$ such that for all $\vec{x} \in \mathbb{Z}_+^d$

$$\operatorname{var}(F_\beta^{\vec{x}}) \leqslant c_2 |\vec{x}| \tag{2.2}$$

Furthermore, if $\mathbb{E}[\exp(tv(e))]$ is finite for some $t>0$, then for all $0<\beta<\infty$ there exists constants $0<C_3, C_4, C_5<\infty$ (depending on β, G and d only) such that for all $\vec{x} \in \mathbb{Z}_+^d$

$$\mathbb{P}\left\{\left|\frac{F_\beta^{\vec{x}}-\mathbb{E}(F_\beta^{\vec{x}})}{\sqrt{|\vec{x}|}}\right| \geqslant u\right\} \leqslant C_3 e^{-C_4 u} \quad \text{for} \quad u \leqslant C_5 |\vec{x}| \tag{2.3}$$

Remark. The above proposition stays true if we replace $F_\beta^{\vec{x}}$ by $F_\beta^{f, n}$ and $|\vec{x}|$ by n.

As pointed out in the introduction, free energy fluctuations are related to transverse displacements of polymer configurations. Our next two propositions establish some rigorous scaling inequalities between χ and ζ. The first one was originally derived by Wehr and Aizenman[11] for the ground state of the directed polymer model, and under somewhat restrictive assumptions on the common distribution of the $v(e)$'s. It was later extended to more general distributions, and to a larger class of models, in ref. 10. Roughly speaking, it establishes that smaller transversal displacements of typical paths imply larger free energy fluctuations. In order to state the result we will have to introduce our versions of the exponents χ and ζ. We start by defining the exponent $\chi_{\hat{x}}(\beta)$ for $\hat{x}$ a unit vector in $\mathbb{R}_+^d$ as

$$\chi_{\hat{x}}(\beta) \equiv \sup\{\gamma \geqslant 0\text{: for some } C>0, \operatorname{var}(F_\beta^{n\hat{x}}) \geqslant C n^{2\gamma} \text{ for all large } n\} \tag{2.4}$$

(here we adopt the convention of computing quantities like $F_\beta^{n\hat{x}}$ at the point in $\mathbb{Z}_+^d$, having as coordinates the integer parts of the corresponding coordinates of $n\hat{x}$). We define in a similar way the exponent χ^f by replacing $F_\beta^{n\hat{x}}$ by $F_\beta^{f, n}$ in the above definition. For a set $A \subset \mathbb{R}_+^d$ and $\vec{z} \in \mathbb{R}_+^d$ define $d(\vec{z}, A) \equiv \inf_{\vec{y} \in A} |\vec{z}-\vec{y}|$, and the "cylinder of radius ρ in the $\hat{x}$ direction" $\mathscr{C}_{\hat{x}}(\rho)$ as

$$\mathscr{C}_{\hat{x}}(\rho) \equiv \{\vec{z} \in \mathbb{R}_+^d : d(\vec{z}, R_{\hat{x}}) \leqslant \rho\} \tag{2.5}$$

where $R_{\hat{x}} = \{\lambda\hat{x}: \lambda \in \mathbb{R}_+\}$. We denote by $\{r \in \mathscr{C}_{\hat{x}}(\rho)\}$ the set of paths starting at the origin and that stay inside $\mathscr{C}_{\hat{x}}(\rho)$. We are now ready to introduce our versions of the exponent ζ. Roughly speaking $\zeta_{\hat{x}}(\beta)$ is such that, with large probability, the measure $\mu_\beta^{n,\hat{x}}$ will be concentrated on directed paths in a cylinder of radius n^γ for all $\gamma > \zeta_{\hat{x}}$. Our precise definition of $\zeta_{\hat{x}}(\beta)$ is the following

$$\zeta_{\hat{x}}(\beta) \equiv \inf\{\gamma > 0: \mu_\beta^{n,\hat{x}}(\{r \in \mathscr{C}_{\hat{x}}(n^\gamma)\}) \xrightarrow{n \to \infty} 1 \text{ in probability}\} \tag{2.6}$$

We define an analogous exponent for the free end case as

$$\zeta^f(\beta) \equiv \inf\{\gamma > 0: \mu_\beta^{f,n}(\{r \in \mathscr{C}_{\hat{x}_{diag}}(n^\gamma)\}) \xrightarrow{n \to \infty} 1 \text{ in probability}\} \tag{2.7}$$

We also define a version of the exponent ζ in the free end case related to the transversal displacement of the *endpoints* $\vec{r}_n$ of polymer configurations as

$$\zeta^{end}(\beta) \equiv \inf\{\gamma > 0: \mu_\beta^{f,n}(\{\vec{r}_n \in \mathscr{C}_{\hat{x}_{diag}}(n^\gamma)\}) \xrightarrow{n \to \infty} 1 \text{ w.p.1}\} \tag{2.8}$$

Notice that the definition of $\zeta^{end}(\beta)$ is in terms of convergence with probability 1 (that is, for (almost) every realization of the random environment), while $\zeta^f(\beta)$ is defined in terms of convergence in probability. The following proposition states that if typical polymer configurations are confined to a cylinder of radius n^ζ then free energy fluctuations are at least of the order of $n^{1-(d-1)\zeta/2}$.

Proposition 2. Under the same hypotheses of Proposition 1 we have for any $0 < \beta < \infty$ and $\hat{x} \in \mathbb{R}_+^d$

$$\chi_{\hat{x}}(\beta) \geqslant \frac{1 - \zeta_{\hat{x}}(\beta)(d-1)}{2} \tag{2.9}$$

Remarks. 1. An analogous result holds if one replaces $\chi_{\hat{x}}$ and $\zeta_{\hat{x}}$ by χ^f and ζ^f respectively.

2. If one assumes the conjectured scaling relation $\chi^f = 2\zeta^f - 1$ then, combined with Proposition 2, it implies $\zeta^f \geqslant 3/(3+d)$ and $\chi^f \geqslant (3-d)/(3+d)$ for all β. In particular one obtains $\zeta^f \geqslant 3/5$ and $\chi^f \geqslant 1/5$ for $d = 2$.

3. In a recent paper Licea *et al.*,[18] using a combination of the zero temperature version of the above inequality and some geometrical arguments, derived the lower bound $\zeta \geqslant 3/5$ (for an appropriate version of the exponent ζ) in the context of the (undirected) first-passage percolation model.

Our next result gives a version (in the form of an inequality) of the relation $\chi = 2\zeta - 1$, a zero temperature version of which was derived in ref. 10. It will involve an additional exponent χ' closely related to χ but which, instead of measuring deviations of $F_\beta^{n\hat{x}}$ from its mean, will refer to deviations of $F_\beta^{n\hat{x}}$ from $nf_\beta(\hat{x})$ where $f_\beta(\hat{x})$ (to be defined precisely later) is the asymptotic free energy per (Euclidean) unit length in the $\hat{x}$ direction. Again we will need some preliminaries. We start with the definition of the function $f_\beta(\vec{x})$ which plays here the same role as the function $\mu(\vec{x})$ in the first-passage percolation model (see ref. 12). Initially, we notice that for a fixed vector $\vec{x} \in \mathbb{Z}_+^d$ one has, for n, m positive integers

$$\mathbb{E}(F_\beta^{(n+m)\,\vec{x}}) \leqslant \mathbb{E}(F_\beta^{n\vec{x}}) + \mathbb{E}(F_\beta^{m\vec{x}}) \tag{2.10}$$

that is, $\{\mathbb{E}(F_\beta^{n\vec{x}})\}_{n \geqslant 1}$ is a subadditive sequence (one easily verifies that $E_{gs}^{n\vec{x}}(\omega) - (|\vec{x}|_1/\beta)\, n \log d \leqslant F_\beta^{n\vec{x}}(\omega) \leqslant E_{gs}^{n\vec{x}}(\omega)$ which, combined with the second moment condition on the potential, guarantees that the expectations in (2.10) are finite). Therefore, it follows from standard arguments that

$$\lim_{n\to\infty} \frac{\mathbb{E}(F_\beta^{n\vec{x}})}{n} \equiv f_\beta(\vec{x}) \tag{2.11}$$

exists and $f_\beta(\vec{x}) = \inf_{n \geqslant 1} \mathbb{E}(F_\beta^{n\vec{x}})/n$ (in particular $f_\beta(\vec{x}) \leqslant \mathbb{E}(F_\beta^{\vec{x}})$). The same procedure can be used to define $f_\beta(\vec{x})$ for all $\vec{x}$ with non-negative rational coordinates by considering limits along sequences where $n\vec{x} \in \mathbb{Z}_+^d$. One can then verify that the function obtained in this manner satisfies, for all $\vec{x}$, $\vec{y}$ having non-negative rational coordinates (see e.g., ref. 8):

(i) $f_\beta(\lambda\vec{x}) = \lambda f_\beta(\vec{x})$ for all non-negative rational λ, and

(ii) $f_\beta(\vec{x} + \vec{y}) \leqslant f_\beta(\vec{x}) + f_\beta(\vec{y})$.

From (i) and (ii) we obtain convexity of $f(\vec{x})$, that is

(iii) $f_\beta(\lambda\vec{x} + (1-\lambda)\,\vec{y}) \leqslant \lambda f_\beta(\vec{x}) + (1-\lambda)\, f_\beta(\vec{y})$ for all rational λ in $[0, 1]$.

From this convexity property it follows that $f_\beta(\vec{x})$ can be extended to a convex function on $\mathbb{R}_+^d$, continuous at interior points of this domain. It turns out that, by invoking Kingman's subadditive ergodic theorem,[16, 17] a stronger result than (2.11) can be obtained. Namely, for every $\vec{x} \in \mathbb{Z}_+^d$

$$\lim_{n\to\infty} \frac{F_\beta^{n\vec{x}}}{n} = f_\beta(\vec{x}) \qquad \text{w.p.1 and in } L^1 \tag{2.12}$$

Our exponent χ' will be associated to deviations of $F_\beta^{\vec{x}}$ from $f_\beta(\vec{x})$. We define it as

$$\chi'(\beta) \equiv \inf\{\gamma > 0\colon |F_\beta^{\vec{x}} - f_\beta(\vec{x})| \leqslant |\vec{x}|^\gamma \text{ for all large } \vec{x} \text{ w.p.1}\} \qquad (2.13)$$

To finish with the preliminaries we discuss a "curvature assumption" that will be part of our hypotheses. For convenience we will assume that $f_\beta(\vec{e}_{diag}) = 1$, which can be done by taking $v(e) \to v(e) + v_0$ for every e and an appropriately chosen (non-random, β dependent) v_0, without affecting thermal averages or sample to sample fluctuations. With this proviso we define the (β dependent) set

$$B_0(\beta) \equiv \{\vec{x}\colon f_\beta(\vec{x}) \leqslant 1\}$$

From $f_\beta(\vec{e}_{diag}) = 1$, the convexity of f_β, and symmetry, it follows that B_0 is a bounded convex subset of $\mathbb{R}^d_+$, symmetric with respect to interchange of coordinates. Notice also that $f_\beta(\hat{e}_1) = \mathbb{E}[v(e)] + v_0$, where $\hat{e}_1$ is a unit vector along the first coordinate axis. Next, for a fixed unit vector $\hat{x} \in \mathbb{R}^d_+$, we consider a hyperplane $T_{\hat{x}}$ at $R_{\hat{x}} \cap \partial B_0 \equiv \vec{O}_{\hat{x}}$ tangent to ∂B_0, with ∂B_0 denoting the boundary of B_0 (if there is more than one we make an arbitrary choice).

Definition. We say that $\hat{x}$ is a direction of curvature for B_0 if there exists a constant $c(\hat{x}, \beta) > 0$, such that for all $\vec{u} \in T_{\hat{x}}$ one has

$$f_\beta(\vec{u}) \geqslant f_\beta(\vec{O}_{\hat{x}}) + c|\vec{u} - \vec{O}_{\hat{x}}|^2$$

The above definition roughly says that $\hat{x}$ is a direction of curvature if one is able to insert a sphere (of sufficiently large radius) between the tangent plane $T_{\hat{x}}$ and the boundary of B_0 at $\vec{O}_{\hat{x}}$. In particular ∂B_0 doesn't have to be smooth, being allowed to have a "corner" in this direction (see also the remark at the end of this section).

Proposition 3. Consider the directed polymer model on $\mathbb{Z}^d_+$ with i.i.d. $v(e)$'s and $\mathbb{E}(v^2(e)) < \infty$.

(a) If $\hat{x}$ is a direction of curvature for $B_0(\beta)$ then

$$\zeta_{\hat{x}}(\beta) \leqslant \frac{1 + \chi'(\beta)}{2} \qquad (2.14)$$

(b) If $\hat{x}_{diag}$ is a direction of curvature for $B_0(\beta)$ then for $* \in \{f, end\}$

$$\zeta^*(\beta) \leqslant \frac{1 + \chi'(\beta)}{2} \qquad (2.15)$$

The questions one would like to address then are on upper bounds for χ' and on what can be said about the "curvature" properties of B_0. In the zero temperature case the extension of the upper bound, $\chi \leqslant 1/2$ to $\chi' \leqslant 1/2$ (under the assumption that $\mathbb{E}[\exp(tv(e))]$ is finite for some $t>0$) follows from the work of Alexander and of Kesten.[8, 9] In that case the bound was obtained by first deriving good (in this case exponential) estimates on the tail of the distribution of $[E_{gs}^{\vec{x}} - \mathbb{E}(E_{gs}^{\vec{x}})]/|\vec{x}|^{\lambda}$ (for $\lambda = 1/2$[9]) and subsequently proving that the above estimate actually implies $\chi' \leqslant \lambda$.[8] For the finite temperature situation we can only go part of the way in establishing this result. The exponential estimate in Proposition 2 and the a-priori bound $\mathbb{E}(F_\beta^{\vec{x}}) \geqslant f_\beta(\vec{x})$ imply that, with probability one, for all $v > 1/2$, $F_\beta^{\vec{x}} - f_\beta(\vec{x}) \geqslant -|\vec{x}|^v$ for $|\vec{x}|$ large enough. To conclude that $\chi' \leqslant 1/2$ one would need an upper bound of the form $\mathbb{E}(F_\beta^{\vec{x}}) - f_\beta(\vec{x}) \leqslant |\vec{x}|^v$ for all $v > 1/2$ and $|\vec{x}|$ large enough (this was obtained, for the zero temperature case, by Alexander[8]). On the other hand an inspection of the proof of Proposition 3 shows that in order to obtain the bounds $\zeta^f \leqslant 3/4$ and $\zeta_{\hat{x}} \leqslant 3/4$ for a given direction of curvature $\hat{x}$, (under the moment condition necessary to obtain the exponential estimate) it would suffice to obtain, for all $\varepsilon > 0$, an upper bound of the form $\mathbb{E}(F^{n\hat{x}}) - nf(\hat{x}) \leqslant c_{12} n^{1/2+\varepsilon}$. At the moment we can only prove such a bound for $\hat{x} = \hat{x}_{diag}$ by an argument due to Sznitman[25] (in the context of Brownian motion in a Poissonian potential).

Proposition 4. Consider the directed polymer model on $\mathbb{Z}_+^d$ with i.i.d., bounded, $v(e)$'s. If $\hat{x}_{diag}$ is a direction of curvature for $B_0(\beta)$ then

$$\text{(a)} \quad \zeta_{\hat{x}_{diag}}(\beta) \leqslant 3/4 \tag{2.16}$$

and, for $* \in \{f, end\}$,

$$\text{(b)} \quad \zeta^*(\beta) \leqslant 3/4 \tag{2.17}$$

As a consequence of Proposition 2 (which shows that upper bounds on ζ imply lower bounds on χ) and Proposition 4 it follows that for $d=2$, if $\hat{x}_{diag}$ is a direction of curvature for $B_0(\beta)$ then $\chi_{\hat{x}_{diag}}(\beta) \geqslant 1/8$, the same lower bound being true for χ^f.

Remark. Our definition of direction of curvature is a natural extension to finite temperature of the definition used in ref. 10 for the zero temperature situation (first-passage percolation). In that case (see ref. 10 for a more complete discussion), it is known[19] that if the probability assigned to the smallest possible value of the potential is above the directed percolation critical value then B_0 has a flat piece and the scaling relation $\chi = 2\zeta - 1$ ceases to hold. In fact in this case one has, along the directions for which

B_0 is flat, the exponents $\chi_{\bar{x}}=0$ and $\zeta_{\bar{x}}=1$ (in their zero temperature version). On the other hand, one expects this to be the only situation where flat pieces will be present. For all other distributions of the potential the relation $\chi=2\zeta-1$ should hold for every direction (except special directions such as the coordinate axis) with a curvature assumption (as defined for example in ref. 10) being satisfied. In the finite temperature situation we expect that due to entropy effects these strictly flat parts should not occur. We notice however that the occurrence of percolation might still have an effect. In fact (as observed in refs. 20, 21, and 22 in the context of the model treated in ref. 14 and 15), the value of $\rho(d)$ obtained in the derivation of (1.9) is given by the probability that the paths of two independent random walkers on $\mathbb{Z}_+^d$, (both starting from the origin and increasing one of their coordinates with equal probability at each step), have an edge in common. If one considers the situation where $v(e)$ assumes the value 0 with probability p then $\mathbb{E}[\exp-2\beta v(e)]/\mathbb{E}^2[\exp-\beta v(e)]\leqslant 1/p$ for all β, therefore if $p>\rho(d)$ there is no strong disorder regime. It also turns out, as observed by Kesten (see Cox and Durrett[24]), that $\rho(d)$ is an upper bound for the oriented percolation critical probability $p_c^{or}(d)$ in d dimensions. It follows that in the situation mentioned above, where there is no strong disorder regime, one has actually the occurrence of oriented percolation of edges with the lowest possible value of the potential. A natural question is if the non-occurrence of a the strong disorder regime persists whenever there is oriented percolation of edges with the lowest possible value of the potential.

3. PROOFS

The proof of the lower bound in Proposition 1 and of Proposition 2 will be based on a general lower bound for variances of random variables which are themselves functions of a collection of random variables having some independence (see Lemma 1 for a precise statement). This bound has been used in similar contexts in refs. 9–11 having been derived in a somewhat abstract setting in ref. 10, where it was used to obtain results for fluctuations in the zero temperature case. Here we will use the same notation and general setting used in that reference. Next, we introduce the notation and state the general lower bound in the form it will be used, and refer to ref. 10 for the details. We remark that the main ingredient (which is also what is used in obtaining the upper bound[11]) is to write the random variable of interest as a sum of martingale differences. The variance of the corresponding sum is then given by the sum of the variances of these martingale differences, and one estimates the variance of each term. We start with a random variable Y with $\mathbb{E}(Y^2)<\infty$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$

where $\Omega = \mathbb{R}^I = \{\omega = (\omega_i : i \in I)\}$ is the space of real valued sequences indexed by a countable index set I, $\mathscr{F} = B^I$ is the usual σ-field generated by Borel cylinder sets and $\mathbb{P}$ is a probability measure on $(\Omega, \mathscr{F})$. We denote by $\mathscr{F}(U)$ with $U \subset I$ the sigma-algebra generated by the random variable π_i, $\pi_i(\omega) = \omega_i$, $i \in U$. In our general setting, we have a sequence U_1, U_2,... of disjoint subsets of I and we express for each k, ω as $(\omega_k, \hat{\omega}_k)$ where ω_k (resp. $\hat{\omega}_k$) is the restriction of ω to U_k (resp. to $I \backslash U_k$). We also have for each k, disjoint events D_k^0 and D_k^1 in $\mathscr{B}^{U_k}$. Define also

$$H_k(\omega) = Y_k^1(\hat{\omega}_k) - Y_k^0(\hat{\omega}_k) \tag{3.1}$$

where

$$Y_k^0(\hat{\omega}_k) = \sup_{\omega_k \in D_k^0} Y((\omega_k, \hat{\omega}_k)), \qquad Y_k^1(\hat{\omega}_k) = \inf_{\omega_k \in D^1{}_k} Y((\omega_k, \hat{\omega}_k)) \tag{3.2}$$

A positive H_k represents a minimum amount that Y is reduced by changing ω_k from D_k^1 to D_k^0 while keeping $\hat{\omega}_k$ fixed.

Lemma 1. Assume the general setting just described and the following three hypotheses about $\mathbb{P}$, the U_k's, the D_k^δ's and Y:

(i) Conditional on $\mathscr{F}(I \backslash \bigcup_k U_k)$, the $\mathscr{F}(U_k)$'s are mutually independent.

(ii) $\exists p, q > 0$ such that for any k,

$$\mathbb{P}(\omega_k \in D_k^0 \mid \mathscr{F}(U_k^c)) \geqslant p, \ \mathbb{P}(\omega_k \in D_k^1 \mid \mathscr{F}(U_k^c)) \geqslant q \quad \text{w.p.1} \tag{3.3}$$

(iii) for every k, $H_k \geqslant 0$ w.p.1.

Then

$$\operatorname{var}(Y) \geqslant pq \sum_k \mathbb{P}(H_k)^2 \tag{3.4}$$

We refer the reader to (the proof of) Theorem 8 in ref. 10 for the proof of this Lemma. We will also make use of the following result about series in the proof of part a) of Proposition 1:

Lemma 2. For any positive a_k's and $m \geqslant 1$

$$\sum_{k=1}^{m} a_k^2 \geqslant \frac{1}{12} \left(\sum_{k=1}^{m} k^{-1} \right)^{-1} \left(\sum_{k=1}^{m-1} k^{-1} \left[k^{-1/2} \sum_{j=1}^{k} a_j \right] \right)^2 \tag{3.5}$$

Proof. See Lemma 2.1 in ref. 10.

Proof of Proposition 1.

(i) Proof of lower bound.

We apply Lemma 1 with I taken as the set of all edges in $\mathbb{Z}^2_+$, $\mathbb{P}$ the product measure which has the joint distribution of the $v(e)$'s, $U_k = \{e_k\}$, where $e_1, e_2,...$ is an ordering of I in which all the edges at distance k from the origin come before those at distance $k+1$, and $Y = F^{\bar{x}}_\beta$. We also introduce the following notation: N_n for the number of edges $e = (\vec{u}, \vec{v})$ for which $|\vec{v}|_1 \leqslant n$, $1_{e_k \in r}$ for the indicator function of the event that edge e_k belongs to the path r and $\langle \eta_k \rangle^{\bar{x}}_\beta \equiv \mu^{\bar{x}}_\beta(1_{e_k \in r})$. The D^δ_k's are taken as

$$D^0_k = (-\infty, \lambda], \qquad D^1_k = [\lambda + \delta\lambda, \infty) \tag{3.6}$$

It follows from Lemma 1 that

$$\operatorname{var}(F^{\bar{x}}_\beta) \geqslant pq \sum_{k=1}^{N_{|\bar{x}|_1}} [\mathbb{E}(H_k)]^2 \tag{3.7}$$

Here $p = \mathbb{P}(v(e) \leqslant \lambda)$, $q = \mathbb{P}(v(e) \geqslant \lambda + \delta\lambda)$, with λ and $\delta\lambda > 0$ chosen such that $p, q > 0$ (which is possible since $\operatorname{var}(v(e)) > 0$) and

$$H_k(\omega) = \inf_{\omega_k \in D^1_k} F^{\bar{x}}_\beta((\omega_k, \hat{\omega}_k)) - \sup_{\omega_k \in D^0_k} F^{\bar{x}}_\beta((\omega_k, \hat{\omega}_k))$$

In order to estimate $H_k(\omega)$ we compute

$$\frac{\partial}{\partial \omega_k} F^{\bar{x}}_\beta = -\frac{1}{\beta} \frac{\partial}{\partial \omega_k} \log Z^{\bar{x}}_\beta = -\frac{1}{\beta} \frac{1}{Z^{\bar{x}}_\beta} \frac{\partial}{\partial \omega_k} Z^{\bar{x}}_\beta$$

and since

$$\begin{aligned} \frac{\partial}{\partial \omega_k} Z^{\bar{x}}_\beta &= \frac{\partial}{\partial \omega_k} \sum_r \exp\left[-\beta \sum_{j=1}^{N_{|\bar{x}|_1}} 1_{e_j \in r} \omega_j \right] \\ &= -\beta \sum_r 1_{e_k \in r} \exp\left[-\beta \sum_{j=1}^{N_{|\bar{x}|_1}} 1_{e_j \in r} \omega_j \right] \end{aligned}$$

we obtain

$$\frac{\partial}{\partial \omega_k} F^{\bar{x}}_\beta((\omega_k, \hat{\omega}_k)) = \mu^{\bar{x}}_\beta(1_{e_k \in r})((\omega_k, \hat{\omega}_k)) \equiv \langle \eta_k \rangle^{\bar{x}}_\beta (\omega_k, \hat{\omega}_k) \geqslant 0 \tag{3.8}$$

Consequently (using $\langle \cdot \rangle$ for $\langle \cdot \rangle^{\bar{x}}_\beta$),

$$H_k(\omega) = \int_\lambda^{\lambda + \delta\lambda} \frac{\partial}{\partial \omega_k} F^{\bar{x}}_\beta(x, \hat{\omega}_k)\, \mathrm{d}x = \int_\lambda^{\lambda + \delta\lambda} \langle \eta_k \rangle (x, \hat{\omega}_k)\, \mathrm{d}x \tag{3.9}$$

From (3.7) we then have

$$\operatorname{var}(F_\beta^{\vec{x}}) \geqslant pq \sum_{k=1}^{N_{|\vec{x}|_1}} \mathbb{E}^2\left[\int_\lambda^{\lambda+\delta\lambda} \langle\eta_k\rangle(x, \hat{\omega}_k)\,\mathrm{d}x\right] \tag{3.10}$$

and, using Lemma 2, we obtain the lower bound

$$\operatorname{var}(F_\beta^{\vec{x}}) \geqslant pq(\delta\lambda)^2 \frac{1}{12}\left(\sum_{k=1}^{N_{|\vec{x}|_1}} k^{-1}\right)^{-1}\left(\sum_{k=1}^{N_{|\vec{x}|_1}-1} k^{-1}\right.$$
$$\left.\left[k^{-1/2}\sum_{j=1}^{k} \mathbb{E}\left[\int_\lambda^{\lambda+\delta\lambda}\langle\eta_j\rangle(x, \hat{\omega}_j)\,\mathrm{d}x\right]\right]\right)^2$$

Now it suffices to show that, for $k \leqslant |\vec{x}|_1$,

$$\sum_{j=1}^{N_k} \mathbb{E}\left[\int_\lambda^{\lambda+\delta\lambda}\langle\eta_j\rangle(x, \hat{\omega}_j)\,\mathrm{d}x\right] \geqslant Dk \tag{3.11}$$

for some $D > 0$. A straightforward calculation gives, for $\omega_k' \geqslant \omega_k$,

$$\frac{\langle\eta_k\rangle(\omega_k', \hat{\omega}_k)}{\langle\eta_k\rangle(\omega_k, \hat{\omega}_k)} = \frac{\exp-(\beta\omega_k')}{\exp-(\beta\omega_k)}\frac{Z(\omega_k, \hat{\omega}_k)}{Z(\omega_k', \hat{\omega}_k)} \geqslant \exp-(\beta\omega_k') \tag{3.12}$$

and also

$$\frac{\partial\langle\eta_k\rangle}{\partial\omega_k} = -\beta\langle\eta_k\rangle[1-\langle\eta_k\rangle] \leqslant 0 \tag{3.13}$$

Thus, $\langle\eta_k\rangle(x, \hat{\omega}_k)$ is decreasing in x and it follows that, for $\omega_k' \leqslant \omega_k$,

$$\langle\eta_k\rangle(\omega_k', \hat{\omega}_k) \geqslant \langle\eta_k\rangle(\omega_k, \hat{\omega}_k)$$

This leads to the estimate

$$\sum_{j=1}^{N_k}\left[\int_\lambda^{\lambda+\delta\lambda}\langle\eta_j\rangle(x, \hat{\omega}_j)\,\mathrm{d}x\right]$$
$$\geqslant \sum_{j=1}^{N_k}\left[\langle\eta_j\rangle(\omega_j, \hat{\omega}_j)\int_\lambda^{\lambda+\delta\lambda}\exp-(\beta x)\,\mathrm{d}x\right]$$
$$= \frac{k}{\beta}\exp-\beta\lambda[1-\exp-\beta(\delta\lambda)] \qquad \text{for} \quad k \leqslant |\vec{x}|_1 \tag{3.14}$$

with the last equality following from the fact that any path from $\vec{0}$ to $\vec{x}$ passes through k edges among the first N_k edges, if $k \leqslant |\vec{x}|_1$. Now, the fact that $N_k = O(k^2)$ immediately implies the desired lower bound for $\mathrm{var}(F_\beta^{\vec{x}})$.

(ii) Proof of upper bound.

We follow here the basic strategy of ref. 9 where it was used to derive an upper bound in the zero temperature case. We start by expressing $\mathrm{var}(F_\beta^{\vec{x}})$ as the sum of the variances of martingale differences. Introducing

$$\Delta_k \equiv [\mathbb{E}(F_\beta^{\vec{x}} \mid \mathscr{G}_k) - \mathbb{E}(F_\beta^{\vec{x}} \mid \mathscr{G}_{k-1})] \tag{3.15}$$

where $\mathscr{G}_k \equiv \mathscr{F}(\{e_1\}) \vee \cdots \vee \mathscr{F}(\{e_k\})$, it follows from a standard calculation that

$$\mathrm{var}(F_\beta^{\vec{x}}) = \sum_{k=1}^{N_{|\vec{x}|_1}} \mathbb{E}(\Delta_k^2) = \sum_{k=1}^{N_{|\vec{x}|_1}} \mathbb{E}[\mathbb{E}(\Delta_k^2 | \mathscr{G}_{k-1})] = \sum_{k=1}^{N_{|\vec{x}|_1}} \mathbb{E}\left[\int \Delta_k^2 \, \mathrm{dG}(\omega_k)\right] \tag{3.16}$$

where, in the last step, we used the independence of the ω_k's, with $\mathrm{G}(\omega_k)$ denoting the distribution of ω_k. It will be convenient in what follows to introduce the i.i.d. random variables σ_k, independent of the ω_k's, and with the same joint distribution as the ω_k's. We can then write Δ_k as (with $\mathbb{E}$ denoting averages with respect to the ω_k's)

$$\Delta_k = \int \mathrm{dG}(\sigma_k)[\mathbb{E}([F_\beta^{\vec{x}}(\omega_k, \hat{\omega}_k) - F_\beta^{\vec{x}}(\sigma_k, \hat{\omega}_k)] \mid \mathscr{G}_k)] \tag{3.17}$$

and, using Cauchy–Schwarz's inequality twice,

$$\Delta_k^2 \leqslant \int \mathrm{dG}(\sigma_k)[\mathbb{E}([F_\beta^{\vec{x}}(\omega_k, \hat{\omega}_k) - F_\beta^{\vec{x}}(\sigma_k, \hat{\omega}_k)]^2 \mid \mathscr{G}_k)] \tag{3.18}$$

Now, from (3.8)

$$\begin{aligned}
&[F_\beta^{\vec{x}}(\omega_k, \hat{\omega}_k) - F_\beta^{\vec{x}}(\sigma_k, \hat{\omega}_k)]^2 \\
&\quad = \left[1_{\sigma_k < \omega_k} \int_{\sigma_k}^{\omega_k} \langle \eta_k \rangle (x, \hat{\omega}_k) \, \mathrm{d}x - 1_{\sigma_k > \omega_k} \int_{\omega_k}^{\sigma_k} \langle \eta_k \rangle (x, \hat{\omega}_k) \, \mathrm{d}x\right]^2 \\
&\quad = \left[1_{\sigma_k < \omega_k} \left(\int_{\sigma_k}^{\omega_k} \langle \eta_k \rangle (x, \hat{\omega}_k) \, \mathrm{d}x\right)^2 + 1_{\sigma_k > \omega_k} \left(\int_{\omega_k}^{\sigma_k} \langle \eta_k \rangle (x, \hat{\omega}_k) \, \mathrm{d}x\right)^2\right]
\end{aligned}$$

and, from the fact that $\langle\eta_k\rangle(x, \hat{\omega}_k)$ is decreasing in x, it follows that

$$\int \Delta_k^2 \, \mathrm{dG}(\omega_k) \leqslant \int \mathrm{dG}(\omega_k) \int \mathrm{dG}(\sigma_k) \, |\omega_k - \sigma_k|^2 \\ \times \mathbb{E}[1_{\sigma_k < \omega_k} \langle\eta_k\rangle^2 (\sigma_k, \hat{\omega}_k) + 1_{\sigma_k > \omega_k} \langle\eta_k\rangle^2 (\omega_k, \hat{\omega}_k) \mid \mathscr{G}_k] \tag{3.19}$$

It is now clear that the integrand on the r.h.s. of the above expression is symmetric in ω_k and σ_k and we obtain

$$\int \Delta_k^2 \, \mathrm{dG}(\omega_k) \leqslant 2\mathbb{E}[\omega_k^2] \, \mathbb{E}[\langle\eta_k\rangle^2 \mid \mathscr{F}_{k-1}] \tag{3.20}$$

Hence, from (3.16) and (3.20), we finally have

$$\begin{aligned} \operatorname{var}(F_\beta^{\vec{x}}) &\leqslant 2\mathbb{E}[v(e)^2] \sum_{k=1}^{N_{|\vec{x}|_1}} \mathbb{E}[\langle\eta_k\rangle^2] \\ &\leqslant 2\mathbb{E}[v(e)^2] \, \mathbb{E}\left[\sum_{k=1}^{N_{|\vec{x}|_1}} \langle\eta_k\rangle\right] = 2\mathbb{E}[v(e)^2] \, |\vec{x}|_1 \end{aligned} \tag{3.21}$$

(since every directed path to $\vec{x}$ contains $|\vec{x}|_1$ edges).

(iii) The exponential estimate in part b) of Proposition 2 follows from a general martingale estimate derived by Kesten (Theorem 3 in ref. 9). In the case where the $v(e)$'s are bounded we can apply the theorem after making the following observations:

(1) The martingale increments Δ_k are given by

$$\Delta_k = \left[\int \mathrm{dG}(\sigma_k) \, \mathbb{E}\left\{ \int_{\sigma_k}^{\omega_k} \langle\eta_k\rangle(x, \hat{\omega}_k) \, \mathrm{d}x \,\middle|\, \mathscr{G}_k \right\} \right]$$

which implies (since the potential is assumed to be bounded)

$$|\Delta_k| \leqslant c \qquad \text{for some} \quad c < \infty$$

(2)

$$\int \Delta_k^2 \, \mathrm{dG}(\omega_k) \leqslant 2\mathbb{E}[v(e)^2] \, \mathbb{E}[\langle\eta_k\rangle^2 \mid \mathscr{G}_{k-1}]$$

(this is just (3.20).

(3)

$$\mathbb{P}\left(\sum_{k=1}^{N_{|\vec{x}|_1}} \langle \eta_k \rangle^2 > u\right) = 0 \qquad \text{for} \quad u > |\vec{x}|_1$$

The hypotheses of Theorem 3 in ref. 9 are implied by 1), 2) and 3) and estimate (2.3) follows for bounded $v(e)$'s. The more general result under the exponential moment condition follows from the same theorem and a truncation argument analogous to the one used in ref. 9.

Proof of Proposition 2. The proof goes as the proof of the lower bound in Proposition 1. With the natural extension of the general setting, introduced in the proof of Proposition 1, from $\mathbb{Z}_+^2$ to $\mathbb{Z}_+^d$ we obtain the estimate

$$\operatorname{var}(F_\beta^{n\hat{x}}) \geqslant pq \sum_{k=1}^{N_{|n\hat{x}|_1}} \mathbb{E}^2\left[\int_\lambda^{\lambda+\delta\lambda} \langle \eta_k \rangle^{n\hat{x}} (x, \hat{\omega}_k)\, dx\right] \tag{3.22}$$

where now $\hat{x}$ is a unit vector in $\mathbb{R}_+^d$ and the sum is over all edges $e = (\vec{u}, \vec{v})$ in $\mathbb{Z}_+^d$ with $|\vec{v}|_1 \leqslant n\, |\hat{x}|_1$. From Cauchy–Schwarz's inequality we have

$$\operatorname{var}(F_\beta^{n\hat{x}}) \geqslant \frac{pq}{|N_{|n\hat{x}|_1} \cap \mathscr{C}_{\hat{x}}(n^\gamma)|}\left[\sum_{e_k \in \mathscr{C}_{\hat{x}}(n^\gamma)} \mathbb{E}\left[\int_\lambda^{\lambda+\delta\lambda} \langle \eta_k \rangle^{n\hat{x}} (x, \hat{\omega}_k)\, dx\right]\right]^2 \tag{3.23}$$

with $|N_{|n\hat{x}|_1} \cap \mathscr{C}_{\hat{x}}(n^\gamma)|$ denoting the numbers edges in $N_{|n\hat{x}|_1} \cap \mathscr{C}_{\hat{x}}(n^\gamma)$. Using the estimate (3.14) we obtain

$$\operatorname{var}(F_\beta^{n\hat{x}}) \geqslant \frac{pq \exp -2\beta\lambda[1 - \exp -\beta(\delta\lambda)]^2}{\beta^2\, |N_{|n\hat{x}|_1} \cap \mathscr{C}_{\hat{x}}(n^\gamma)|}\left[\sum_{e_k \in \mathscr{C}_{\hat{x}}(n^\gamma)} \mathbb{E}[\langle \eta_k \rangle^{n\hat{x}} (\omega_k, \hat{\omega}_k)]\right]^2 \tag{3.24}$$

Taking $\gamma = \zeta_{\hat{x}} + \varepsilon$ with $\varepsilon > 0$ it follows, from the definition of $\zeta_{\hat{x}}$, that $\mathbb{E}[\sum_{e_k \in \mathscr{C}_{\hat{x}}(n^\gamma)} \langle \eta_k \rangle (\omega_k, \hat{\omega}_k)] \geqslant c_3 n$ for all n large enough. Inequality (2.9) is now a consequence of the definition of the exponents $\chi_{\hat{x}}$ and the fact that $|N_{|n\hat{x}|_1} \cap \mathscr{C}_{\hat{x}}(n^{\gamma(d-1)})| = O(n^{1+\gamma(d-1)})$. The result for the free end case follows from a similar argument and we omit the details.

Proof of Proposition 3. The constants c_i appearing in the proof of this proposition will all be positive and finite, and may depend on β, d, $\hat{x}$ and G.

Proof of part (a).

We start by picking an edge $e_k = (\vec{u}, \vec{v})$ belonging to a path from $\vec{0}$ to $n\vec{O}_{\hat{x}} \equiv \vec{x}_n$ with $\hat{x}$ a direction of curvature for B_0. Define also

$$F_\beta^{\vec{x}, \vec{y}} = -\frac{1}{\beta} \log Z_\beta^{\vec{x}, \vec{y}} \tag{3.25}$$

with

$$Z_\beta^{\vec{x}, \vec{y}}(\omega) \equiv \sum_{r: \vec{x} \to \vec{y}} \exp[-\beta E(r; \omega)] \tag{3.26}$$

Then (omitting the index β from now on)

$$\langle \eta_k \rangle_\beta^{\vec{x}_n} = e^{-\beta v(e_k)} \frac{Z^{\vec{0}, \vec{u}} Z^{\vec{v}, \vec{x}_n}}{Z^{\vec{0}, \vec{x}_n}} \leqslant \frac{Z^{\vec{0}, \vec{u}} Z^{\vec{u}, \vec{x}_n}}{Z^{\vec{0}, \vec{x}_n}} \tag{3.27}$$

Next, for $v > 0$, consider the event

$$D_v(\vec{u}) = \{ |F^{\vec{x}, \vec{y}} - f(\vec{y} - \vec{x})| \leqslant |\vec{x} - \vec{y}|^v \text{ for } (\vec{x}, \vec{y}) = (\vec{0}, \vec{u}), (\vec{u}, \vec{x}_n) \text{ and } (\vec{0}, \vec{x}_n) \} \tag{3.28}$$

From (3.27) it follows that, on $D_v(\vec{u})$,

$$\begin{aligned} \langle \eta_k \rangle_\beta^{\vec{x}_n} &\leqslant \exp[-\beta(f(\vec{u}) + f(\vec{x}_n - \vec{u}) - f(\vec{x}_n))] \exp \beta[|\vec{u}|^v + |\vec{x}_n - \vec{u}|^v + |\vec{x}_n|^v] \\ &\leqslant \exp -(\beta \delta_f(\vec{u})) \exp 3\beta |\vec{x}_n|^v \end{aligned} \tag{3.29}$$

with $\delta_f(\vec{u}) = f(\vec{u}) + f(\vec{x}_n - \vec{u}) - f(\vec{x}_n)$. Notice that here we used only a lower bound for $Z^{\vec{0}, \vec{x}_n}$. We now write $\vec{u}$ as $\vec{u} = u(\vec{O}_{\hat{x}} + \vec{\Delta}_{\vec{u}})$ with $\vec{O}_{\hat{x}} + \vec{\Delta}_{\vec{u}} \in T_{\hat{x}}$ and $u < n$. From the fact that $\hat{x}$ is a direction of curvature and $f(\lambda \vec{x}) = \lambda f(\vec{x})$ we have

$$f(\vec{u}) = u f(\vec{O}_{\hat{x}} + \vec{\Delta}_{\vec{u}}) \geqslant u f(\vec{O}_{\hat{x}}) + c_4 u |\vec{\Delta}_{\vec{u}}|^2 \tag{3.30}$$

and

$$f(\vec{x}_n - \vec{u}) = f((n-u) \vec{O}_{\hat{x}} - u\vec{\Delta}_{\vec{u}}) \geqslant (n-u) f(\vec{O}_{\hat{x}}) + c_4 \frac{u^2}{(n-u)} |\vec{\Delta}_{\vec{u}}|^2 \tag{3.31}$$

Therefore,

$$\delta_f(\vec{u}) = c_4 \left(1 + \frac{u}{(n-u)}\right) u |\vec{\Delta}_{\vec{u}}|^2 \geqslant c_4 u |\vec{\Delta}_{\vec{u}}|^2 \tag{3.32}$$

Now, if $\vec{u} \notin \mathscr{C}_{\bar{x}}(n^{\gamma})$ and $\gamma > 0$, it follows, from a simple geometrical consideration, that $|u\vec{\Delta}_{\vec{u}}| \geqslant c_5 n^{\gamma}$ and consequently $\delta_f(\vec{u}) \geqslant c_6(n^{2\gamma}/u) \geqslant c_7 n^{2\gamma-1}$. Substituting in (3.33) we get

$$\begin{aligned}\langle \eta_k \rangle_{\beta}^{\vec{x}_n} &\leqslant \exp(-\beta c_7 n^{2\gamma-1}) \exp 3\beta\, |\vec{x}_n|^{\nu} \leqslant \exp -\beta(c_7 n^{2\gamma-1} - c_8 n^{\nu}) \\ &\leqslant c_9 \exp -(\beta c_{10} n^{2\gamma-1}) \qquad \text{if} \quad 2\gamma - 1 > \nu \end{aligned} \tag{3.33}$$

Finally, the definition of χ' implies that, for $2\gamma - 1 > \nu > \chi'$, $\lim_{n\to\infty} \mathbb{P}(\bigcap_{\vec{u} \notin \mathscr{C}_{\bar{x}}(n^{2\gamma-1})} D_{\nu}(\vec{u})) = 1$ which, combined with (3.33) gives, for $2\gamma - 1 > \chi'$

$$\mathbb{P}\left(\sum_{e_k \notin \mathscr{C}_{\bar{x}}(n^{2\gamma-1})} \langle \eta_k \rangle_{\beta}^{\vec{x}_n} \leqslant c_{11} n^d \exp -(\beta c_{10} n^{2\gamma-1})\right) \to 1 \tag{3.34}$$

as $n \to \infty$. Part (a) of Proposition 3 now follows.

Proof of part (b).

The proof is along the same lines as the proof of part (a) and we give here only a brief sketch for the case $* = end$. Again we start by picking an edge $e_k = (\vec{u}, \vec{v})$ with $|\vec{v}|_1 = n$. Introducing

$$H_n = \{\vec{x} \in \mathbb{Z}_+^d : |\vec{x}|_1 = n\}$$

one has

$$\langle \eta_k \rangle_{\beta}^{f,n} = \frac{Z^{\vec{0}, \vec{v}}}{\sum_{\vec{x} \in H_n} Z^{\vec{0}, \vec{x}}} \leqslant \frac{Z^{\vec{0}, \vec{v}}}{Z^{\vec{0}, (n/d)\, \vec{e}_{diag}}} \tag{3.35}$$

Defining, for $\nu > 0$, the event

$$\begin{aligned} D_{\nu}(\vec{u}) = \{ |F^{\vec{x}, \vec{y}} - f(\vec{y} - \vec{x})| \leqslant |\vec{x} - \vec{y}|^{\nu} \\ \text{for } (\vec{x}, \vec{y}) = (\vec{0}, \vec{u}) \text{ and } (\vec{0}, (n/d)\, \vec{e}_{diag})\} \end{aligned} \tag{3.36}$$

one has for $2\gamma - 1 > \nu > \chi'$, $\lim_{n\to\infty} 1_{[\bigcap_{\vec{u} \notin \mathscr{C}_{\bar{x}}(n^{2\gamma-1})} D_{\nu}(\vec{u})]} = 1$ w.p.1. (recalling that now we are considering only edges of the form $e_k = (\vec{u}, \vec{v}$ with $|\vec{v}|_1 = n)$). By an argument similar to the one described in the proof of part (a) one obtains

$$\sum_{e_k \notin \mathscr{C}_{\bar{x}}(n^{2\gamma-1})} \langle \eta_k \rangle_{\beta}^{f,n} \to 0 \tag{3.37}$$

w.p.1. as $n \to \infty$ and part (b) for the case $* = end$ follows.

Proof of Proposition 4. In view of the remarks following the statement of Proposition 3 it suffices to prove that, for every $\varepsilon > 0$, $\mathbb{E}(F^{m\hat{x}}) - mf(\hat{x}) \leqslant c_{12} m^{1/2+\varepsilon}$ with $\hat{x} = \hat{x}_{diag}$. We follow the same strategy as the proof of Theorem 3.1. in ref. 25 (see also ref. 26 for a similar type of argument in the first-passage percolation context). In order to simplify the notation we will take here $\beta = 1$ since the factors of β play no role and can be easily recovered if needed. As in the proof of Proposition 3 we consider

$$H_n = \{\vec{x} \in \mathbb{Z}_+^d : |\vec{x}|_1 = n\}$$

and define for positive integers m

$$g_\lambda(m) = \log \sum_{\vec{x} \in H_m} \mathbb{E}[\exp(-\lambda F^{\vec{0}, \vec{x}})]$$

Taking now $\vec{x} \in H_{n+m}$ it follows that

$$Z^{\vec{0}, \vec{x}} = \sum_{\vec{y} \in H_m} Z^{\vec{0}, \vec{y}} Z^{\vec{y}, \vec{x}} \leqslant \sum_{\vec{y}, \vec{y}' \in H_m} Z^{\vec{0}, \vec{y}} Z^{\vec{y}', \vec{x}}$$

Consequently,

$$\exp(-F^{\vec{0}, \vec{x}}) \leqslant \sum_{\vec{y} \in H_m} [\exp(-F^{\vec{0}, \vec{y}})] \sum_{\vec{y}' \in H_m} [\exp(-F^{\vec{y}', \vec{x}})] \tag{3.38}$$

Raising both sides to the power λ, with $0 < \lambda \leqslant 1$, and averaging over the $v(e)$'s (using the fact that for all $\vec{y}, \vec{y}' \in H_m$, $F^{\vec{0}, \vec{y}}$ and $F^{\vec{y}', \vec{x}}$ are independent random variables) one obtains

$$\mathbb{E}[\exp(-\lambda F^{\vec{0}, \vec{x}})] \leqslant \mathbb{E}\left[\sum_{\vec{y} \in H_m} \exp(-\lambda F^{\vec{0}, \vec{y}})\right] \mathbb{E}\left[\sum_{\vec{y}' \in H_m} \exp(-\lambda F^{\vec{y}', \vec{x}})\right]$$

Now, summing over $\hat{x} \in H_{n+m}$ (the number of terms in the sum is of order $(m+n)^{d-1}$) and taking logarithms one obtains the subadditive relation

$$g_\lambda(m+n) \leqslant g_\lambda(m) + g_\lambda(n) + r(m+n)$$

with $r(m) = c_{13} \log m$. It then follows from ref. 27 that $G_\lambda = \lim_{m \to \infty} [g_\lambda(m)/m]$ exists and

$$G_\lambda \leqslant \frac{g_\lambda(m)}{m} - \frac{r(m)}{m} + 4 \sum_{k=2m}^{\infty} \frac{r(k)}{k(k+1)} \leqslant \frac{g_\lambda(m)}{m} + c_{14} \frac{1 + \log m}{m} \tag{3.39}$$

for all $m \geqslant 1$. From the estimate

$$g_\lambda(m) \geqslant \log \mathbb{E}[\exp\{-\lambda F^{\vec{0},(m/d)\vec{e}_{diag}}] \geqslant -\lambda \mathbb{E}(F^{\vec{0},(m/d)\vec{e}_{diag}}) \tag{3.40}$$

we also obtain

$$G_\lambda \geqslant -\lambda f(\vec{e}_{diag}/d) \tag{3.41}$$

Therefore, combining (3.39) and (3.41) we have

$$-\lambda f(\vec{e}_{diag}/d) \leqslant \frac{g_\lambda(m)}{m} + c_{14}\frac{1+\log m}{m} \tag{3.42}$$

We now derive the upper bound for $g_\lambda(m)$ in terms of $\mathbb{E}[F^{\vec{0},2(m/d)\vec{e}_{diag}}]$ which allow us to obtain a bound on $\mathbb{E}[F^{\vec{0},(m/d)\vec{e}_{diag}}] - mf(\vec{e}_{diag}/d)$. We start by writing $g_\lambda(m)$ as

$$g_\lambda(m) = \log \sum_{\vec{x}\in H_m} \exp(-\lambda\mathbb{E}(F^{\vec{0},\vec{x}}))\, \mathbb{E}[\exp \lambda\{\mathbb{E}(F^{\vec{0},\vec{x}}) - F^{\vec{0},\vec{x}}\}] \tag{3.43}$$

Observing now that for every $\vec{x} \in H_m$ one has $Z^{\vec{0},(2m/d)\vec{e}_{diag}} \geqslant Z^{\vec{0},\vec{x}} Z^{\vec{x},(2m/d)\vec{e}_{diag}}$ and therefore $\mathbb{E}[F^{\vec{0},(2m/d)\vec{e}_{diag}}] \leqslant 2\mathbb{E}[F^{\vec{0},\vec{x}}]$ we get

$$\begin{aligned} g_\lambda(m) \leqslant & -\frac{\lambda}{2}\mathbb{E}[F^{\vec{0},(2m/d)\vec{e}_{diag}}] \\ & + \log[c_{16} m^{(d-1)} \sup_{\vec{x}\in H_m} \mathbb{E}[\exp \lambda\{\mathbb{E}(F^{\vec{0},\vec{x}}) - F^{\vec{0},\vec{x}}\}]] \end{aligned} \tag{3.44}$$

Combining this with (3.42) then gives

$$\begin{aligned} &\mathbb{E}[F^{\vec{0},(2m/d)\vec{e}_{diag}}] - 2mf(\vec{e}_{diag}/d) \\ &\quad \leqslant c_{17}\frac{(1+\log m)}{\lambda} + \frac{1}{\lambda}\log \sup_{\vec{x}\in H_m} \mathbb{E}[\exp \lambda\{\mathbb{E}(F^{\vec{0},\vec{x}}) - F^{\vec{0},\vec{x}}\}] \end{aligned} \tag{3.45}$$

The above inequality is valid for every $m \geqslant 1$ and every $\lambda \in (0, 1]$. In particular we can choose λ depending on m. To finish the proof it suffices to show that for every $\varepsilon > 0$ if we take $\lambda = m^{-(1/2+\varepsilon)}$ then $\sup_m \sup_{\vec{x}\in H_m} \mathbb{E}[\exp \lambda\{\mathbb{E}(F^{\vec{0},\vec{x}}) - F^{\vec{0},\vec{x}}\}] < \infty$. This is now a consequence of the exponential estimate in part b) of Proposition 1 and the fact that for bounded $v(e)$'s $|F_\beta^{\vec{0},\vec{x}} - \mathbb{E}(F_\beta^{\vec{0},\vec{x}})| \leqslant c_{18}|\vec{x}|$. In this case (2.3) is then also valid for $u > c_{18}\sqrt{|\vec{x}|}$. Therefore, (2.3) is valid for all u. This in turn implies the desired uniform bound on $\mathbb{E}[\exp(|\vec{x}|^{-(1/2+\varepsilon)}\{\mathbb{E}(F^{\vec{0},\vec{x}}) - F^{\vec{0},\vec{x}}\})]$.

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REFERENCES

1. D. S. Fisher and D. A. Huse, Directed paths in a random potential, *Phys. Rev. B* **43**:10728 (1991).
2. D. A. Huse and C. L. Henley, Pinning and roughening of domain walls in Ising systems due to random impurities, *Phys. Rev. Lett.* **54**:2708–2711 (1985).
3. D. A. Huse, C. L. Henley, and D. S. Fisher, *Phys. Rev. Lett.* **55**:2924 (1985).
4. T. Hwa and D. S. Fisher, Anomalous fluctuations of directed polymers in random media, *Phys. Rev. B.* **49**:3136 (1994).
5. M. Kardar, G. Parisi, and Y.-C. Zhang, Dynamic scaling of growing interfaces, *Phys. Rev. Lett.* **56**:889–892 (1986).
6. M. Kardar, Roughening by impurities at finite temperatures, *Phys. Rev. Lett.* **55**:2923 (1985).
7. M. Kardar and Y.-C. Zhang, Scaling of directed polymers in random media, *Phys. Rev. Lett.* **58**:2087–2090 (1987).
8. K. Alexander, Approximation of subadditive functions and convergence rates in limiting-shape results, *Ann. Probab.* (to appear).
9. H. Kesten, On the speed of convergence in first passage percolation, *Ann. Appl. Probab.* **3**:296–338 (1993).
10. C. M. Newman and M. S. T. Piza, Divergence of shape fluctuations in two dimensions, *Ann. Probab.* **23**:977–1005 (1995).
11. J. Wehr and M. Aizenman, Fluctuations of extensive functions of quenched random couplings, *J. Stat. Phys.* **60**:287–306 (1990).
12. H. Kesten, Aspects of first passage percolation. In *Lecture Notes in Mathematics*, vol. 1180 125–264, Springer-Verlag.
13. J. Krug and H. Spohn, Kinetic roughening of growing surfaces. In *Solids far from equilibrium: growth, morphology and defects*. Edited by C. Godrèche. 479–582. Cambridge University Press.
14. J. Z. Imbrie and T. Spencer, Diffusion of directed polymers in a random environment, *J. Stat. Phys.* **52**:608–626 (1988).
15. E. Bolthausen, A note on the diffusion of directed polymers in a random environment, *Comm. Math. Phys.* **123**:529–534 (1989).
16. J. F. C. Kingman, The ergodic theory of subadditive stochastic processes, *J. Roy. Stat. Soc. B.* **30**:499–510 (1968).
17. T. M. Liggett, An improved subadditive ergodic theorem, *Ann. Probab.* **13**:1279–1285 (1985).
18. C. Licea, C. M. Newman, and M. S. T. Piza, Superdiffusivity in first-passage percolation, *Probab. Theory Relat. Fields* **106**:559–591 (1996).
19. R. Durrett and T. M. Liggett, The shape of the limit set in Richardson's growth model, *Ann. Probab.* **9**:186–193 (1981).
20. R. Song and X. Y. Zhou, A remark on diffusion of directed polymers in random environments, *J. Stat. Phys.* 277–289 (1996).

21. P. Olsen and R. Song, Diffusion of directed polymers in a strong random environment, *J. Stat. Phys.* **83**:727–738 (1996).
22. Y. G. Sinai, A remark concerning random walks in random potentials, *Fund. Math.* **147**:173–180 (1995).
23. J. T. Cox and R. Durret, Oriented percolation in dimensions $d \geqslant 4$: bounds and asymptotic formulas, *Math. Proc. Camb. Phil Soc.* **93**:151–162 (1983).
24. J. Cook and B. Derrida, Disordered hierarchical lattices, *J. Stat. Phys.* **57**:89–139 (1989).
25. A. S. Sznitman, Distance fluctuations and Lyapunov exponents, *Ann. Probab.* **24**:1507–1530 (1996).
26. K. Alexander, A note on some rates of convergence in first-passage percolation, *Ann. Appl. Probab.* **3**:81–90 (1993).
27. J. M. Hammersley, Generalization of the fundamental theorem on subadditive functions, *Proc. Camb. Phil. Soc.* **58**:235–238 (1961).
28. M. A. Moore, T. Blum, J. P. Doherty, M. Marsili, J.-P. Bouchaud, and P. Claudin, Glassy solutions of the Kardar-Parisi-Zhang equation, *Phys. Rev. Lett.* **74**:4257–4260 (1995).
29. H. Kinzelbach and M. Lassig, Interacting flux lines in a random medium, *Phys. Rev. Lett.* **75**:2208–2211 (1995).